Using induction to prove a DFA recognizes a Language

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Consider the DFA $M_1 = (Q, \Sigma, \delta, q_0, F)$ where

$$\begin{aligned} Q &= \{q_1, q_2\} \\ \Sigma &= \{0, 1\} \\ q_0 &= q_1 \\ F &= \{q_2\} \end{aligned}$$

and $\delta : Q \times \Sigma \to Q$ defined by the following table

δ	0	1
q_1	q_1	q_2
q_2	q_2	q_1

Confirm that the formal description is equivalent to the state diagram in Fig. 1

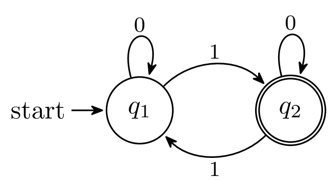

Figure 1: A state diagram for M_1

Definition 1. *Consider $w \in \Sigma^*$, $q_i, q_j \in Q$, and let $n = |w|$. $q_i \xrightarrow{w}_M q_j$ iff $\exists r_0, \ldots, r_n \in Q$ such that*

1. $r_0 = q_i$
2. $r_i = \delta(r_{i-1}, w_i)$, $\forall 1 \leq i \leq n$
3. $r_n = q_j$

Statement 1. *Let $A = \{w \in \Sigma^* \mid w$ has an odd number of 1s$\}$. $A = L(M_1)$.*

Let's start by proving the forward direction: $A \subseteq L(M)$.

To prove this, we'll start with a useful and more straightforward lemma, following our intuitions that a string with an odd number of ones will swap the state we're in, and one with an even number will keep us in the same state. Formally, that looks like:

Lemma 1. *Consider $w \in \Sigma^*$. $q_2 \xrightarrow{w}_{M_1} q_2$ if the number of 1s in w is even, and $q_1 \xrightarrow{w}_{M_1} q_2$ if the number of 1s in w is odd.*

Proof. We will proceed by induction over the length of w!

Base Case: Consider $w = \varepsilon$ (length 0). ε contains an even number of 1s (0), so we must show that $q_1 \xrightarrow{\varepsilon}_{M_1} q_1$. Consider the "sequence" of states $r_0 = q_1$. Property 2 is trivially true, since we only have one state, and properties 1 and 3 are self-evident.

Inductive Step: We must show that if statement 1 is true for all $w \in \Sigma^*$ with $|w| = n - 1$, it is true for all $w \in \Sigma^*$ with $|w| = n$.

Consider an arbitrary $w = w_1 \dots w_n \in \Sigma^*$ with $|w| = n$. Let $w' = w_2 \dots w_n$. Since $|w'| = n - 1$, our inductive hypothesis applies, and thus we can assume that if the number of 1s in w' is even, $q_2 \xrightarrow{w}_{M_1} q_2$, and if not, $q_1 \xrightarrow{w}_{M_1} q_2$.

Now, since we know that $w = w_1 w'$, we can break into cases on the identity of w_1:

Case 0: $w_1 = 0$. If $w_1 = 0$, then w and w' have the same number of 1s. Let's prove each half of the statement in turn:

Assume that w has an even number of 0s. Then w' has an even number of 1s, and we can assume that $q_2 \xrightarrow{w'}_{M_1} q_2$. This means that there exists $r_0, \dots r_{n-1}$ such that

$$
\begin{aligned}
r_0 &= r_{n-1} = q_2 \\
r_i &= \delta(r_{i-1}, w'_i) \\
&= \delta(r_{i-1}, w_{i+1})
\end{aligned}
$$

Then construct a new sequence $r'_0, \dots r'_n$ such that

$$
\begin{aligned}
r'_0 &= q_2 \\
r'_i &= r_{i-1} \forall 1 \leq i \leq n
\end{aligned}
$$

That is, we build $r'_0, \dots, r_n$ by prepending q_2 to $r_0, \dots, r_{n-1}$. Then observe that we have constructed our sequence such that $r'_0 = r'_n = q_2$, leaving us to prove that $r_i = \delta(r_{i-1}, w_i)$, $\forall 1 \leq i \leq n$. For $i \geq 2$, this follows from our inductive hypothesis, because the sequence is simply $r_0, \dots, r_{n-1}$. To show it's true for $i = 1$, observe that since $w_1 = 0$ and $r_0 = q_2$, $delta(r_0, w_1) = q_2 = r_2$, allowing us to conclude that $q_2 \xrightarrow{w}_{M_1} q_2$.

A similar argument lets us show the other half. Assume w has an odd number of 0s. By our inductive hypothesis, we learn that $q_1 \xrightarrow{w'}_{M_1} q_2$ since it has the same number of 1s and w' has length $n - 1$, which means there exists $r_0, \dots r_{n-1}$ such that

$$
\begin{aligned}
r_0 &= q_1 \\
r_{n-1} &= q_2 \\
r_i &= \delta(r_{i-1}, w'_i) \\
&= \delta(r_{i-1}, w_{i+1})
\end{aligned}
$$

Again, construct $r'_0, \dots r'_n$ by prepending q_2:

$$r'_0 = q_1$$
$$r'_i = r_{i-1}, \forall 1 \leq i \leq n$$

And obser that $r'_0 = q_1$ as necessary, $r'_n = r_{n-1} = q_2$, as necessary, leaving us only to show that $r_i = \delta(r_{i-1}, w_i)$, $\forall 1 \leq i \leq n$. Again, this follows from our inductive hypothesis for all cases but $i = 1$, because the sequence is simply $r_0, \dots, r_{n-1}$. To show it's true for $i = 1$, observe that since $w_1 = 0$ and $r_0 = q_1$, $delta(r_0, w_1) = q_1 = r_2$, allowing us to conclude that $q_2 \xrightarrow{w}_{M_1} q_2$.

Case 1: Assume $w_1 = 1$. We've done it in excruciating detail for Case 0, so I'll be more terse in this case.

Again, let $w' = w_2 \dots w_n$ and observe that w' is subject to our inductive hypothesis.

Now, we should observe that is w contains an odd number of 1s, w' contains an even number and vice versa. Let's now show both halves of the statement in turn:

Assume w has an odd number of 1s. This means w' has an even number of 1s, and so we can apply our inductive hypothesis to assume $q_2 \xrightarrow{w'}_{M_1} q_2$, which means there exists $r_0, \dots r_{n-1}$ with the appropriate properties. Now construct $r_0, \dots r_n$ by prepending q_1:

$$r'_0 = q_1$$
$$r'_i = r_{i-1}, \forall 1 \leq i \leq n$$

And observe that $r_0 = q_1$ and $r'_n = q_2$, and, like in the other cases, $r'_i = \delta(r'_{i-1}, w_i)$ for $2 \leq i \leq n$ by the inductive hypothesis and the $i = 1$ case follows from the definition of δ and our assumption that $w_1 = 1$. □

Now we can get to the proof of Statement 1.

Proof (Statement 1). First, we show $A \subseteq L(M_1)$.

Suppose $w \in A$, and thus the number of 1s in w is odd. Thus, by lemma, we know $q_1 \xrightarrow{w} q_2$, and thus there exists $r_0, \dots, r_n$ such that

$$r_0 = q_1 = q_0$$
$$r_n = q_2 \in F$$
$$r_i = \delta(r_{i-1}, w_i), \forall 1 \leq i \leq n$$

Thus, by definition, M_1 accepts w, and thus $w \in L(M)$, as desired.

Then we need to show that $L(M_1) \subseteq A$. We must show that if M_1 accepts w, w contains an odd number of 1s. It'll be easier to proceed via *contrapositive* for this direction: If w contains an even number of 1s, then M_1 will not accept w. That is, if w contains an even number of 1s, there does *not* exist a sequence $r_0, \dots, r_n$ that has the 3 properties in the definition of acceptance. Of course, the first 2 properties force our hand: If such an $r_0, \dots, r_n$ exists, then $r_0 = q_0 = q_1$ and $r_i = \delta(r_{i-1}, w_i)$ for all $1 \leq i \leq n$, and such a sequence surely exists by the definition of our DFA.

Thus, to prove w is rejected, we must then prove that $r_n \notin F$. Since there is only 1 state not in F, this is equivalent to proving that $q_1 \xrightarrow{w}_{M_1} q_1$ if w contains an even number of 1s. I'll leave completing that proof as an exercise: It's nearly identical to Lemma 1! □